

OPTIMAL APPROXIMATION AND KOLMOGOROV WIDTHS ESTIMATES FOR CERTAIN SINGULAR CLASSES RELATED TO EQUATIONS OF MATHEMATICAL PHYSICS

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Abstract. Solutions of numerous equations of mathematical physics such as elliptic, weakly singular, singular, hypersingular integral equations belong to functional classes $\bar{Q}_{r\gamma}^u(\Omega, 1)$ and $Q_{r\gamma}^u(\Omega, 1)$ defined over l -dimensional hypercube $\Omega = [-1, 1]^l, l = 1, 2, \dots$. The derivatives of classes' representatives grow indefinitely when the argument approaches the boundary $\partial\Omega$. In this paper we estimate the Kolmogorov and Babenko widths of two functional classes $\bar{Q}_{r\gamma}^u(\Omega, 1)$ and $Q_{r\gamma}^u(\Omega, 1)$. We construct local splines belonging to those classes, such that the errors of approximation are of the same order as that of the estimated widths. Thus we construct optimal with respect to order methods for approximating the functional classes $\bar{Q}_{r\gamma}^u(\Omega, 1)$ and $Q_{r\gamma}^u(\Omega, 1)$. One can use these results for constructing methods optimal with respect to order for approximating a unit ball of the Sobolev spaces with logarithmic and polynomial weights.

Key words. Kolmogorov widths; Babenko widths; optimal approximation; splines.

1. Introduction. Let B be a Banach space, $X \subset B$ be a compact set, and $\Pi : X \rightarrow R_n$ be a mapping of $X \subset B$ onto a finite-dimensional space R_n .

DEFINITION 1.1. Lorentz (1986). Let L^n be n -dimensional subspaces of the linear space B . The Kolmogorov width $d_n(X, B)$ is defined by

$$d_n(X, B) = \inf_{L^n} \sup_{x \in X} \inf_{u \in L^n} \|x - u\|, \quad (1.1)$$

where the outer infimum is calculated over all n -dimensional subspaces of L^n .

DEFINITION 1.2. (Anuchina et al., 1979; Babenko, 1985). The Babenko width $\delta_n(X)$ is defined by $\delta_n(X) = \inf_{\Pi: X \rightarrow R^n} \sup_{x \in X} \text{diam} \Pi^{-1}\Pi(x)$, where the infimum is calculated over all continuous mappings $\Pi : X \rightarrow R^n$.

If the infimum in (1.1) is attained for some L^n , this subspace is called an extremal subspace.

The widths evaluation for various spaces of functions play an important role in numerical analysis and approximation theory since this problem is closely related to many optimality problems such as ϵ -complexity of integration and approximation, optimal differentiation, and optimal approximation of solutions of operator equations.

For a detailed study of these problems in view of the general theory of optimal algorithms we refer to Traub et al. (1980).

Kolmogoroff (1936) formulated the problem of evaluating the widths $d_n(X, B)$, the discovery of extremal subspaces of L^n . Kolmogoroff (1936) also evaluated $d_n(X, B)$ for certain compact sets X . Kolmogorov asserted to determine the exact value of $d_n(X, B)$ because it might lead to the discovery of extremal subspaces, and therefore to new and better methods of approximation. Anuchina et al. (1979); Babenko (1985) promoted using extremal subspaces of compacts X in constructing numerical methods in physics and mechanics.

The most general results were obtained in estimating the Kolmogorov widths in Sobolev spaces W_p^r on unit balls $B(W_p^r)$. Stechkin (1954) estimated the widths $d_n(B(W_1^r), L_2)$ and $d_n(B(W_\infty^r), L_\infty)$. Tikhomirov (1960) obtained the exact values

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of $d_n(B(W_1^r), C)$. The widths $d_n(B(W_p^r), L_q)$ for various p and q were studied by several authors, e.g. Gluskin (1974), Ismagilov (1974, 1977), Maiorov (1975), Makovoz (1972), Kashin (1977). Maiorov (1975) and Kashin (1977) obtained the final estimates of $d_n(B(W_p^r), L_q)$ for $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. The widths of various classes of multivariable functions were analyzed by several scientists, e.g., Anuchina et al. (1979), Babenko (1985, 1971), Boykov (2007), Hollig (1980), Kudryavtsev (1995), Lorentz et al. (1996), Temlyakov (1986), where the books Anuchina et al. (1979), Boykov (2007), Lorentz (1986), Lorentz et al. (1996), Temlyakov (1986), and the articles Babenko (1985); Kudryavtsev (1995) may serve as reviews.

Solutions of numerous problems of analysis, mechanics, electrodynamics, and geophysics lead to the necessity to develop optimal methods for approximating special classes of functions. The classes $Q_{r\gamma}(\Omega, 1)$ consist of functions having bounded derivatives up to order r in a closed domain Ω and higher order derivatives in $\Omega \setminus \delta\Omega$, whose modulus increases unboundedly in a neighbourhood of the boundary $\partial\Omega$ (see Definitions 2.1 – 2.5). The classes $Q_{r\gamma}(\Omega, 1)$ describe solutions of elliptic equations Babenko (1985), weakly singular, singular, and hypersingular integral equations Boykov (2004).

The functions representable by singular and hypersingular integrals with moving singularities $\int_{-1}^1 \frac{\varphi(\tau)}{\tau-t} d\tau$, $t \in (-1, 1)$; $\int_{-1}^1 \frac{\varphi(\tau)}{(\tau-t)^p} d\tau$, $t \in [-1, 1]$, $p = 2, 3, \dots$; $\int_{-1}^1 \int_{-1}^1 \frac{\varphi(\tau_1, \tau_2)}{((\tau_1-t_1)^2 + (\tau_2-t_2)^2)^{p/2}} d\tau_1 d\tau_2$, $t_1, t_2 \in [-1, 1]$, $p = 3, 4, \dots$ also belong to $Q_{r\gamma}(\Omega, 1)$ (see Boykov, 2005, 2009).

Apparently Babenko (1985) defined the class of functions $Q_r(\Omega, 1)$ to emphasize its role in construction approximation in numerous important problems in mathematical physics.

The relationship between the functional class $Q_r(\Omega, 1)$ (as well as $Q_{r\gamma}(\Omega, 1)$) and compacts in the weighted Sobolev space $W_\infty^r(\Omega, 1, \rho)$ follows from the definition of the classes.

Let $\Omega = [-1, 1]^l$, $l = 1, 2, \dots$, $t = (t_1, \dots, t_l)$, $r = (r_1, \dots, r_l)$, $r_i, i = 1, \dots, l$, be integers. Let $\rho = d(t, \Gamma)$ be the l_∞ -distance between a point t and the boundary $\delta\Omega$.

The class $W_\infty^r(\Omega, 1, \rho)$ consists of functions $f \in C(\Omega)$, which have bounded partial derivatives of orders $i = 0, 1, \dots, r$ in Ω and partial derivatives of orders $i = r + 1, \dots, 2r + 1$ in $\Omega \setminus \delta\Omega$ with the norm $\|f\| = \|f\|_{L_\infty(\Omega)} + \sum_{i=1}^r \sum_{i=i_1+\dots+i_l} \|\partial^i f / \partial t_1^{i_1} \dots \partial t_l^{i_l}\|_{L_\infty(\Omega)} + \sum_{i=r+1}^{2r+1} \sum_{i=i_1+\dots+i_l} \|(\rho(t))^{i-r} \partial^i f / \partial t_1^{i_1} \dots \partial t_l^{i_l}\|_{L_\infty(\Omega)} \leq 1$, where i_j are nonnegative integers, $0 \leq i_j \leq i$, $j = 1, \dots, l$, $i = i_1 + \dots + i_l$.

Similarly one can define the classes of functions $W_\infty^{r\gamma}(\Omega, 1, \rho)$, $W_\infty^{r\gamma u}(\Omega, 1, \rho)$, and $\bar{W}_\infty^{r\gamma u}(\Omega, 1, \rho)$ which are counterparts of the classes $Q_{r\gamma}(\Omega, 1)$, $Q_{r\gamma}^u(\Omega, 1)$, and $\bar{Q}_{r\gamma}^u(\Omega, 1)$.

The results of this paper can be extended to the classes $W_\infty^r(\Omega, 1, \rho)$, $W_\infty^{r\gamma}(\Omega, 1, \rho)$, $W_\infty^{r\gamma u}(\Omega, 1, \rho)$, and $\bar{W}_\infty^{r\gamma u}(\Omega, 1, \rho)$.

The widths estimates for the sets of functions $B(W_\infty^r(\Omega, 1, \rho))$, $B(W_\infty^{r\gamma}(\Omega, 1, \rho))$, $B(W_\infty^{r\gamma u}(\Omega, 1, \rho))$, and $B(\bar{W}_\infty^{r\gamma}(\Omega, 1, \rho))$ are of interest since they play an important role in various applied problems, for example, problems of hydrodynamics. The author intends to use the obtained results in his further works in constructing optimal numerical methods for solving some problems of mathematical physics.

2. Definitions of the function classes and previous results. Babenko (1985) defined the class $Q_r(\Omega, 1)$ (Definition 2.1) and declared the problem of estimating the Kolmogorov and Babenko widths of $Q_r(\Omega, 1)$ to be one of the most

important problems in the numerical analysis. Later on this problem was solved by the author (see Boykov, 1987, 1998).

The classes $Q_{r\gamma}(\Omega, 1)$, $Q_{r\gamma p}(\Omega, 1)$, and $B_{r\gamma}(\Omega, 1)$ generalize the class $Q_r(\Omega, 1)$. Boykov (1998, 2007) estimated the Kolmogorov and Babenko widths and constructed local splines for approximation of functions from $Q_{r\gamma}(\Omega, 1)$, $Q_{r\gamma p}(\Omega, 1)$, and $B_{r\gamma}(\Omega, 1)$. The error of approximation obtained by local splines has the same order as that of the corresponding values of the Kolmogorov and Babenko widths. Below we list the definitions of the functional classes $Q_{r\gamma}(\Omega, 1)$, $Q_{r\gamma p}(\Omega, 1)$, $Q_{r\gamma}^u(\Omega, 1)$, and $\bar{Q}_{r\gamma}^u(\Omega, 1)$.

Let $\Omega = [-1, 1]^l$, $l \geq 1$, $\Gamma = \partial\Omega$ be the boundary of Ω , and u, r be positive integers. Let $t = (t_1, \dots, t_l)$, $v = (v_1, \dots, v_l)$, $|v| = v_1 + \dots + v_l$, $D^v = \partial^{|v|}/\partial t_1^{v_1} \dots \partial t_l^{v_l}$ and v_i be nonnegative integers, $i = 1, 2, \dots, l$.

DEFINITION 2.1. (Babenko, 1985). Let $\Omega = [-1, 1]^l$, $l = 1, 2, \dots$. The class $Q_r(\Omega, 1)$ consists of functions $f \in C^r(\Omega)$ satisfying $\max_{t \in \Omega} |D^v f(t)| \leq 1$, $0 \leq |v| \leq r$, $|D^v f(t)| \leq (d(t, \Gamma))^{-(|v|-r)}$, $t \in \Omega \setminus \Gamma$, $r < |v| \leq 2r + 1$, where $d(t, \Gamma)$ is the l_∞ -distance between a point t and Γ .

DEFINITION 2.2. (Boikov, 1998, 2007). Let $\Omega = [-1, 1]^l$, $l = 1, 2, \dots$. The class $Q_{r\gamma}(\Omega, 1)$ consists of functions $f \in C^r(\Omega)$ satisfying $\max_{t \in \Omega} |D^v f(t)| \leq 1$, $0 \leq |v| \leq r$, $|D^v f(t)| \leq (d(t, \Gamma))^{-(|v|-r-\zeta)}$, $t \in \Omega \setminus \Gamma$, $r < |v| \leq s$. Note, $s = r + [\gamma]$, $\zeta = [\gamma] - \gamma$.

DEFINITION 2.3. (Boikov, 1998, 2007). Let $\Omega = [-1, 1]^l$, $l = 1, 2, \dots$. The class $Q_{r\gamma p}(\Omega, 1)$ consists of functions $f \in C^r(\Omega)$ satisfying $\max_{t \in \Omega} |D^v f(t)| \leq 1$, $0 \leq |v| \leq r$, $\int_{\Omega} |(d(t, \Gamma))^\gamma D^v f(t)|^p dt \leq 1$, $r < |v| \leq s$, where $1 \leq p < \infty$, $v = v_1 + \dots + v_l$, $0 \leq v_i \leq s$, $i = 1, 2, \dots, l$, $s = r + [\gamma]$, $\zeta = [\gamma] - \gamma$.

DEFINITION 2.4. Let $\Omega = [-1, 1]^l$, $l = 1, 2, \dots$. Let γ and u be positive integers. The class $\bar{Q}_{r\gamma}^u(\Omega, 1)$ consists of functions $f \in C^{r-1}(\Omega)$ satisfying $\max_{t \in \Omega} |D^v f(t)| \leq 1$, $0 \leq |v| \leq r - 1$, $|D^v f(t)| \leq (1 + |\ln^u d(t, \Gamma)|)$, $t \in \Omega \setminus \Gamma$, $|v| = r$, $|D^v f(t)| \leq (1 + |\ln^{u-1} d(t, \Gamma)|)/(d(t, \Gamma))^{|v|-r}$, $t \in \Omega \setminus \Gamma$, $r < |v| \leq s$, where $s = r + \gamma$.

DEFINITION 2.5. Let $\Omega = [-1, 1]^l$, $l = 1, 2, \dots$. Let u be a positive integer, and γ be a non-integer. The class $Q_{r\gamma}^u(\Omega, 1)$ consists of functions $f \in C^r(\Omega)$ satisfying $\max_{t \in \Omega} |D^v f(t)| \leq 1$, $0 \leq |v| \leq r$, $|D^v f(t)| \leq (1 + |\ln^u d(t, \Gamma)|)/(d(t, \Gamma))^{|v|-r-\zeta}$, $r < |v| \leq s$, $t \in \Omega \setminus \Gamma$, where $s = r + [\gamma]$, $\zeta = [\gamma] - \gamma$.

DEFINITION 2.6. Let $G = [a, b]$. The class $W^r(1)$, $r = 1, 2, \dots$, consists of functions $f \in C[a, b]$ which have absolutely continuous derivatives of orders $j = 0, 1, \dots, r - 1$ and a piecewise continuous derivative $f^{(r)}$ satisfying $|f^{(r)}| \leq 1$.

DEFINITION 2.7. Let $G = [a_1, b_1; \dots; a_l, b_l] = [a_1, b_1] \times \dots \times [a_l, b_l]$, $l = 2, 3, \dots$. The class $C_l^r(1)$, $r = 1, 2, \dots$ consists of functions $f \in C[G]$ which have absolutely continuous partial derivatives of orders $j = 0, 1, \dots, r - 1$ and a piecewise continuous partial derivative of order r satisfying $\max_{t \in G} |D^r f(t)| \leq 1$.

Now we briefly describe the notations we use throughout this paper.

Let $f \in W^r(1)$, $t \in [a, b]$, $c \in [a, b]$. We denote by $T_{r-1}(f, [a, b], c)$ the Taylor polynomial of f of order $r - 1$ with respect to the point c , i.e. $T_{r-1}(f, [a, b], c) = \sum_{j=0}^{r-1} (f^{(j)}(c)/j!)(t - c)^j$. For $f(t_1, t_2) \in C_2^r(1)$, $t = (t_1, t_2) \in G = [a_1, b_1; a_2, b_2]$, $v = (v_1, v_2) \in G$ we can re-write its Taylor polynomial of order $r - 1$ as $T_{r-1}(f, G, v) = \sum_{j=0}^{r-1} d_j(f, v)/j!$, where $d_j(f, v)$ is a polynomial of order j given by $d_j(f, v)(t) =$

$\sum_{i=0}^j C_j^i (\partial^j f(v) / \partial t_1^i \partial t_2^{j-i}) (t_1 - v_1)^i (t_2 - v_2)^{j-i}$. The Taylor polynomials for functions of $l \geq 2$ variables can be re-written similarly.

Many types of differential and integral equations such as elliptic equations (Babenko, 1985), weakly singular integral equations (Vainikko, 1991), singular, and hypersingular integral equations (Boykov, 2009; Boykov et al., 2010) have solutions that belong to the functional sets similar to $\bar{Q}_{r\gamma}^u(\Omega, 1)$, and $Q_{r\gamma}^u(\Omega, 1)$. Therefore evaluating the widths of $\bar{Q}_{r\gamma}^u(\Omega, 1)$, and $Q_{r\gamma}^u(\Omega, 1)$ and finding the extremal subspaces are important problems in numerical analysis.

The author (Boykov, 1987, 1998, 2007) developed optimal with respect to order to accuracy methods for approximating the classes $Q_{r\gamma}(\Omega, 1)$ and $B_{r\gamma}(\Omega, 1)$. Afterwards, these methods were used in developing (optimal with respect to order to accuracy and complexity) approximate methods for solving Fredholm and Volterra weakly singular integral equations (Boykov, 2004), in estimating an accuracy of elliptic equations solutions (Boykov, 2003), and in developing optimal with respect to accuracy cubature rules for evaluating many-dimensional integrals (Boykov, 1989). Solutions of some classes of weakly singular, singular, and hypersingular integral equations belong to the classes $\bar{Q}_{r\gamma}^u(\Omega, 1)$ and $Q_{r\gamma}^u(\Omega, 1)$.

In this paper we obtain the weak asymptotic estimates of the Kolmogorov and Babenko widths of the classes $\bar{Q}_{r\gamma}^u(\Omega, 1)$ and $Q_{r\gamma}^u(\Omega, 1)$. We also construct local splines, which yield the optimal order of approximation (in the sense of the widths d_n and δ_n) for functions in $\bar{Q}_{r\gamma}^u(\Omega, 1)$, $Q_{r\gamma}^u(\Omega, 1)$.

The author intends to use the optimal methods for approximating the functional classes $\bar{Q}_{r\gamma}^u(\Omega, 1)$ and $Q_{r\gamma}^u(\Omega, 1)$ proposed in this paper for developing optimal methods of solving weakly singular, singular, and hypersingular integral equations.

We start with recalling the following well-known assertions.

LEMMA 2.8. (Lorentz, 1986). Let D be a Hausdorff space, $X \subset C(D)$. There exist $n+1$ points t_i , $i = 0, \dots, n$ and a number $\epsilon \geq 0$ with the following property: For each distribution of signs $\lambda_i = \pm 1$, $i = 0, \dots, n$, there is a function $f_0 \in X$ such that $\text{sign } f_0(t_i) = \lambda_i$, $|f_0(t_i)| \geq \epsilon$, $i = 0, \dots, n$. Then $d_n(X, C) \geq \epsilon$ in the space $C(D)$.

LEMMA 2.9. (Anuchina et al., 1979). Let B be a Banach space, $X \subset B$ be a compact set. The inequality $\delta_n(X) \leq 2d_n(X, B)$ is true.

Let ζ_k , $k = 1, 2, \dots, r$, be the zeros of the Chebyshev polynomial of the first kind of degree r . Moreover, let f be a continuous function on $[-1, 1]$, i.e. $f \in C([-1, 1])$. Finally, denote by $L_r(f, [-1, 1])$ the interpolating polynomial with respect to the Chebyshev nodes ζ_1, \dots, ζ_r .

LEMMA 2.10. (Goncharov, 1954). If $f \in W^r$, $r = 1, 2, \dots$, then $\|f - L_r(f, [-1, 1])\|_{C[-1, 1]} \leq \|f^{(r)}\|_{C[-1, 1]} 1/(r!2^{r-1})$. Now we briefly recall the estimates of the Babenko and Kolmogorov widths of the classes $Q_{r\gamma}(\Omega, 1)$, and $Q_{r\gamma p}(\Omega, 1)$ with respect to C and L_q .

If in the following theorems no restrictions on one or several of the parameters are given, then the full range as described in the respective definitions is admissible.

THEOREM 2.11. (Boykov, 1998, 2007). Let $\Omega = [-1, 1]$. Then $\delta_n(Q_{r\gamma}(\Omega, 1)) \asymp d_n(Q_{r\gamma}(\Omega, 1), C) \asymp n^{-s}$.

THEOREM 2.12. (Boykov, 1998, 2007). Let $\Omega = [-1, 1]$. If γ is integer, then

$$d_n(Q_{r\gamma p}(\Omega, 1), L_q) \asymp \begin{cases} n^{-s+1/p-1/q}, & 1 \leq p < q \leq 2, \\ n^{-s+1/p-1/2}, & 1 \leq p \leq 2, \ 2 < q < \infty, \\ n^{-s}, & 1 \leq q \leq p < \infty, \ 2 \leq p \leq q < \infty. \end{cases}$$

THEOREM 2.13. (Boykov, 1998, 2007). Let $\Omega = [-1, 1]^l$, $l \geq 2$. Then

$$\delta_n(Q_{r\gamma}(\Omega, 1)) \asymp d_n(Q_{r\gamma}(\Omega, 1), C) \asymp \begin{cases} n^{-(s-\gamma)/(l-1)}, & v > l/(l-1), \\ n^{-s/l}, & v < l/(l-1), \\ n^{-s/l}(\ln n)^{s/l}, & v = l/(l-1), \end{cases}$$

where $v = s/(s - \gamma)$.

THEOREM 2.14. (Boykov, 1998, 2007). Let $\Omega = [-1, 1]^l$, $l \geq 2$, $1 \leq q \leq p < \infty$. If γ is integer, then

$$d_n(Q_{r\gamma p}(\Omega, 1), L_q) \asymp \begin{cases} n^{-r/(l-1)}, & v > l/(l-1), \\ n^{-s/l}, & v < l/(l-1), \\ (\ln n/n)^{s/l}, & v = l/(l-1), \end{cases}$$

where $v = s/(s - \gamma)$.

THEOREM 2.15. (Boykov, 1998, 2007). Let $\Omega = [-1, 1]^l$, $l \geq 2$, $1 \leq p < q \leq 2$. If γ is integer, then

$$d_n(Q_{r\gamma p}(\Omega, 1), L_q) \asymp \begin{cases} n^{-(r-l/p+l/q)/(l-1)}, & v > l/(l-1), \\ n^{-(s-l/p+l/q)/l}, & v < l/(l-1), \\ (\ln n/n)^{(s-l/p+l/q)/l}, & v = l/(l-1), \end{cases}$$

where $v = (s - l/p + l/q)/(s - l/p + l/q - \gamma)$.

THEOREM 2.16. (Boykov, 1998, 2007). Let $\Omega = [-1, 1]^l$, $l \geq 2$, $1 \leq p \leq 2$, $2 < q < \infty$. If γ is integer, then

$$d_n(Q_{r\gamma p}(\Omega, 1), L_q) \asymp \begin{cases} n^{-(r-l/p+l/q)/(l-1)+1/q-1/2}, & v > l/(l-1), \\ n^{-(s-l/p+l/q)/l}, & v < l/(l-1), \end{cases}$$

where $v = (s - l/p + l/q)/(s - l/p + l/q - \gamma)$.

Remark. The articles (Boykov, 1998, 2007) also contain explicit constructions of local splines which yield the optimal order of approximation error in Theorems 2.11–2.16. Hence these splines can be regarded as optimal methods of approximation in the sense of the Kolmogorov and Babenko widths.

In this paper we extend some of these results to the classes $\bar{Q}_{r\gamma}^u([-1, 1]^l, 1)$ and $Q_{r\gamma}^u([-1, 1]^l, 1)$, $l \geq 1$.

3. Widths of the classes $\bar{Q}_{r\gamma}^u([-1, 1], 1)$ and $Q_{r\gamma}^u([-1, 1], 1)$ of functions of one variable. In this section we estimate the Kolmogorov and Babenko widths for each of the functional classes $\bar{Q}_{r\gamma}^u(\Omega, 1)$ and $Q_{r\gamma}^u(\Omega, 1)$, $\Omega = [-1, 1]$.

THEOREM 3.1. Let $\Omega = [-1, 1]$. Let r, u, γ be positive integers, $s = r + \gamma$. Then $\delta_n(\bar{Q}_{r\gamma}^u(\Omega, 1)) \asymp d_n(\bar{Q}_{r\gamma}^u(\Omega, 1), C) \asymp n^{-s}$.

Proof. First we estimate the infimum for $\delta_n(\bar{Q}_{r\gamma}^u(\Omega, 1))$. Note $Q_{r\gamma}(\Omega, 1) \subset \bar{Q}_{r\gamma}^u(\Omega, 1)$. By Theorem 2.11 we know $\delta_n(Q_{r\gamma}(\Omega, 1)) \asymp n^{-s}$. Therefore

$$\delta_n(\bar{Q}_{r\gamma}^u(\Omega, 1)) \geq \delta_n(Q_{r\gamma}(\Omega, 1)) \asymp n^{-s}. \quad (3.1)$$

To construct a continuous local spline with n parameters that approximates the functions of $\bar{Q}_{r\gamma}^u(\Omega, 1)$ with the accuracy An^{-s} , we study two cases: i) $u = 1$, and ii) $u > 1$.

i). Let $u = 1$. We divide the interval $[-1, 1]$ into $2N$ subintervals by the points $t_k = -1 + (k/N)^v$ and $\tau_k = 1 - (k/N)^v$, $k = 0, 1, \dots, N$, $v = s/(s - \gamma)$. Then divide the obtained interval $[t_0, t_1]$ into M ($M = \lceil \ln N \rceil$) subintervals by the points $t_{0,j} = t_0 + (t_1 - t_0)j/M$, $j = 0, 1, \dots, M$. Continuing the partition procedure we subdivide $[\tau_1, \tau_0]$ into M subintervals by $\tau_{0,j} = \tau_0 - (\tau_0 - \tau_1)j/M$, $j = 0, 1, \dots, M$.

For each interval $[a, b]$ we now choose a polynomial $P_s(f, [a, b])$ interpolating $f(t) \in \bar{Q}_{r\gamma}^u(\Omega, 1)$ at the endpoints a and b in the following way. Denote the zeros of the Chebyshev polynomial of the first kind of degree s by ζ_k , $k = 1, 2, \dots, s$. Map $[\zeta_1, \zeta_s] \subset [-1, 1]$ with an affine-linear transformation onto $[a, b]$ so that the points ζ_1 and ζ_s are mapped onto a and b respectively. Denote the images of the points ζ_i under this mapping by ζ'_i , $i = 1, 2, \dots, s$. We then denote by $P_s(f, [a, b])$ the interpolation polynomial of degree $s - 1$ with respect to the nodes ζ'_i , $i = 1, 2, \dots, s$.

Now define the function f_N on $[-1, 1]$ to piecewise consist of the polynomials $P_s(f, [t_{0,j}, t_{0,j+1}])$, $P_s(f, [t_k, t_{k+1}])$, $P_s(f, [\tau_{k+1}, \tau_k])$, $P_s(f, [\tau_{0,j+1}, \tau_{0,j}])$, $j = 0, 1, \dots, M - 1$, $k = 1, 2, \dots, N - 1$.

With these definitions in hand, we can now estimate the pointwise approximation error $\|f - f_N\|$. On the intervals $\Delta_k^1 = [t_k, t_{k+1}]$, $k = 1, 2, \dots, N - 1$ we obtain

$$\|f - f_N\|_{C(\Delta_k^1)} \leq E_{s-1}(f, \Delta_k^1)(1 + \lambda_s) \leq c\|f - T_{s-1}(f, \Delta_k^1, t_k)\|\lambda_s \leq \frac{c(t_{k+1} - t_k)^s}{(k/N)^{v\gamma}s!} = cN^{-s}, \quad (3.2)$$

where λ_s is the Lebesgue constant with respect to the nodes used for $P_s(f, \Delta_k^1)$, $E_s(f, [a, b])$ is the best approximation of the function f by polynomials of degree at most s in the norm of $C[a, b]$,

Similar estimates are true in $\Delta_k^2 = [\tau_{k+1}, \tau_k]$, $k = 1, 2, \dots, N - 1$.

Throughout this paper, we denote the constants that do not depend on N by c .

Now let $k = 0$. It is well-known that it holds $\|f - f_N\|_{C(\Delta_{0,0}^1)} \leq E_{s-1}(f, \Delta_{0,0}^1)(1 + \lambda_s)$,

where $\Delta_{0,0}^1 = [t_{0,0}, t_{0,1}]$. Here λ_s is the Lebesgue constant with respect to the nodes used for $P_s(f, \Delta_{0,0}^1)$.

Using Taylor's expansion $T_{r-1}(f, \Delta_{0,0}^1, -1)$ with the remainder in integral form we find

$$\begin{aligned} E_{s-1}(f, \Delta_{0,0}^1) &\leq \|f - T_{r-1}(f, \Delta_{0,0}^1, -1)\|_{C(\Delta_{0,0}^1)} \leq \frac{1}{(r-1)!} \max_{t \in \Delta_{0,0}^1} \left| \int_{-1}^t f^{(r)}(\tau)(t-\tau)^{r-1} d\tau \right| \leq \\ &\leq \frac{1}{(r-1)!} \max_{t \in \Delta_{0,0}^1} \int_{-1}^t (1 + |\ln(1+\tau)|)(t-\tau)^{r-1} d\tau \leq \frac{c}{r!} (h_{0,0}^r \ln h_{0,0} + h_{0,0}^r) \leq c \frac{1}{N^s \ln^{r-1} N}, \end{aligned}$$

where $h_{0,k} = |t_{0,k+1} - t_{0,k}|$, $k = 0, 1, \dots, M - 1$.

Therefore, $E_{s-1}(f, \Delta_{0,0}^1) \leq cN^{-s} \ln^{r-1} N$.

Since the Lebesgue constant λ_s is independent of N due to its invariance under rescaling we finally obtain

$$\|f - f_N\|_{C(\Delta_{0,0}^1)} \leq c/(N^s \ln^{r-1} N). \quad (3.3)$$

One can obtain a similar estimate in $\Delta_{0,0}^2 = [\tau_{0,1}, \tau_{0,0}]$.

With similar arguments we obtain on $\Delta_{0,j}^1 = [t_{0,j}, t_{0,j+1}]$, $1 \leq j \leq M-1$

$$\|f - f_N\|_{C(\Delta_{0,j}^1)} \leq \frac{c\lambda_s}{(1+t_{0,j})^\gamma} h_{0,j}^s \leq \frac{c\lambda_s}{j^\gamma} (N^v \ln N)^\gamma \left(\frac{1}{N^v \ln N} \right)^s \leq \frac{c}{N^s \ln^r N}. \quad (3.4)$$

Similar estimates are true in $\Delta_{0,j}^2 = [\tau_{0,j+1}, \tau_{0,j}]$, $j = 1, 2, \dots, M-1$.

The total number n of nodes used in constructing the local spline equals $n = 2((\lfloor nN \rfloor - 1)(s-1) + (N-1)(s-1)) + 1$. Therefore $N \sim n/(2(s-1))$.

It follows from (3.2) – (3.4) that we have constructed a continuous local spline f_N which approximates $f \in \bar{Q}_{r\gamma}^1(\Omega, 1)$ with the accuracy $O(n^{-s})$. From Definition 1.1, it follows that

$$d_n(\bar{Q}_{r\gamma}^1(\Omega, 1), C) \leq cn^{-s}. \quad (3.5)$$

Using Lemma 2.9 and the inequalities (3.1), (3.5) we complete the proof of the theorem for $u = 1$. We now consider

ii) $u \geq 2$. We use the same points t_k and τ_k and intervals Δ_k^1 and Δ_k^2 as before. Additionally put $M_0 = \lceil \ln^{u/r} N \rceil$, and $M_k = \lceil \ln^{(u-1)/s}(N/k) \rceil$, $k = 1, \dots, N-1$. Divide each Δ_k^1 and Δ_k^2 into M_k , $k = 0, 1, \dots, N-1$, equal subintervals and denote the latter ones by $\Delta_{k,j}^i$, $i = 1, 2$, $j = 0, 1, \dots, M_k - 1$, $k = 0, 1, \dots, N-1$.

We shall approximate $f \in \bar{Q}_{r\gamma}^u(\Omega, 1)$ within each $\Delta_{k,j}^i$ by the interpolating polynomial $P_s(f, \Delta_{k,j}^i)$, $i = 1, 2$, $j = 0, 1, \dots, M_k - 1$, $k = 0, 1, \dots, N-1$. Let f_N be the spline composed of the polynomials $P_s(f, \Delta_{k,j}^i)$.

Starting with approximating the error in $\Delta_{0,0}^i$, we have in $\Delta_{0,0}^i$ $\|f - f_N\|_{C(\Delta_{0,0}^i)} \leq E_{s-1}(f, \Delta_{0,0}^1)(1 + \lambda_s)$.

Using Taylor's expansion $T_{r-1}(f, \Delta_{0,0}^1, -1)$ we find

$$\begin{aligned} E_{s-1}(f, \Delta_{0,0}^1) &\leq \|f - T_{r-1}(f, \Delta_{0,0}^1, -1)\|_{C(\Delta_{0,0}^1)} \leq \frac{1}{(r-1)!} \max_{t \in \Delta_{0,0}^1} \left| \int_1^t f^{(r)}(\tau)(t-\tau)^{r-1} d\tau \right| \\ &\leq \frac{1}{(r-1)!} \max_{t \in \Delta_{0,0}^1} \left| \int_1^t (1 + |\ln^u(1+\tau)|)(t-\tau)^{r-1} d\tau \right| \leq ch_{00}^r |\ln^u h_{00}| \leq \\ &\leq c \left(\frac{1}{N^v M_0} \right)^r \left| \ln^u \left(\frac{1}{N^v M_0} \right) \right| \leq c \frac{1}{N^s \ln^u N} (\ln^u N + \ln^u \ln N) \leq c \frac{1}{N^s}, \end{aligned}$$

where $h_{00} = h_0/M_0$, $h_0 = t_1 - t_0$.

Finally, $\|f - f_N\|_{C(\Delta_0^i)} \leq cN^{-s}$ in Δ_0^i , $i = 1, 2$.

Our next step is to obtain the estimate of $\|f - f_N\|_{C(\Delta_{k,j}^i)}$, $j = 0, 1, \dots, M_k - 1$, $i = 1, 2$, $k = 1, 2, \dots, N-1$. It is obvious that

$$\begin{aligned} \|f - f_N\|_{C(\Delta_{k,1}^1)} &\leq \frac{c(t_{k,1} - t_{k,0})^s}{s!} \left(\frac{N}{k} \right)^{v\gamma} \left(1 + \left| \ln^{u-1} \left(\frac{N}{k} \right) \right| \right) \\ &\leq c \left(\frac{h_k}{M_k} \right)^s \left(\frac{N}{k} \right)^{v\gamma} \left(1 + \ln^{u-1} \frac{N}{k} \right) \leq \end{aligned}$$

$$\begin{aligned}
&\leq c \left(\left(\left(\frac{k+1}{N} \right)^v - \left(\frac{k}{N} \right)^v \right) \frac{1}{\left(\ln \frac{N}{k} \right)^{(u-1)/s}} \right)^s \left(\frac{N}{k} \right)^{v\gamma} \left(1 + \ln^{u-1} \frac{N}{k} \right) \\
&\leq c \frac{(k+\theta)^{(v-1)s-v\gamma}}{N^s} \leq \frac{c}{N^s},
\end{aligned}$$

where $t_{k,j} = t_k + (t_{k+1} - t_k)j/M_k$, $j = 0, 1, \dots, M_k$, $k = 0, 1, \dots, N-1$.

The errors $\|f - f_N\|_{C(\Delta_{k,j}^1)}$, $\|f - f_N\|_{C(\Delta_{k,j}^2)}$, $j = 1, 2, \dots, M_k - 1$, $k = 1, 2, \dots, N-1$, can be estimated similarly.

Combining the estimates obtained above we have $\|f - f_N\|_{C([-1,1])} \leq cN^{-s}$.

Next we need to estimate the number of nodes used to construct the local spline f_N . For this purpose, we first estimate the number m of subintervals $\Delta_{k,j}^i$, $i = 1, 2$, $j = 0, 1, \dots, M_k - 1$, $k = 0, 1, \dots, N-1$.

Let $q = (u-1)/s$. Then

$$\begin{aligned}
m &= 2 \sum_{k=0}^{N-1} M_k \leq 2 \left(\ln^{\frac{u}{r}} N + \sum_{k=1}^{N-1} \ln^{\frac{u-1}{s}} \frac{N}{k} \right) \leq c \left(N + \sum_{k=2}^{N-1} \ln^q \frac{N}{k} \right) \\
&\leq c \left(N + N \int_1^N \frac{\ln^q t}{t^2} dt \right) \leq cN.
\end{aligned}$$

Therefore the total number of nodes used in constructing f_N equals to $n = (s-1)m + 1 = cN$. Hence, $\|f - f_N\|_{C([-1,1])} \leq cN^{-s} \leq cn^{-s}$.

Thus, $d_n(\bar{Q}_{r\gamma}^u(1), C) \leq cn^{-s}$.

Comparing the preceding inequality to the estimate (3.1) we complete the proof of the theorem for $u \geq 2$. \square

THEOREM 3.2. Let $\Omega = [-1, 1]$. Let r, u be positive integers, and γ be a positive non-integer. The estimate $\delta_n(Q_{r\gamma}^u(\Omega, 1)) \asymp d_n(Q_{r\gamma}^u(\Omega, 1), C) \asymp n^{-s}$ holds, where as usual $s = r + \lceil \gamma \rceil$.

Proof. First, we estimate the infimum for $\delta_n(Q_{r\gamma}^u(\Omega, 1))$. Note $Q_{r\gamma}(\Omega, 1) \subset Q_{r\gamma}^u(\Omega, 1)$. By Theorem 2.11, the inequality $\delta_n(Q_{r\gamma}(\Omega, 1)) \geq cn^{-s}$ holds. Therefore $\delta_n(Q_{r\gamma}^u(\Omega, 1)) \geq cn^{-s}$. Next, we construct a local spline f_N with at most cn nodes approximating the given function $f \in Q_{r\gamma}^u(\Omega, 1)$ with error at most cn^{-s} .

We use the same subdivision procedure into intervals $\Delta_{k,j}^1 = [t_{k,j}, t_{k,j+1}]$ and $\Delta_{k,j}^2 = [\tau_{k,j+1}, \tau_{k,j}]$ as in the proof of Theorem 3.1. Here $M_0 = \lceil \ln^{u/(r+1-\mu)} N \rceil$, $M_k = \lceil \ln^{u/s} N \rceil$, $k = 1, \dots, N-1$, $\mu = 1 - \zeta$.

We approximate a function f on $[-1, 1]$ by the spline f_N which piecewise consists of the polynomials $P_s(f, \Delta_{k,j}^i)$, $i = 1, 2$, $j = 0, 1, \dots, M_k - 1$, $k = 0, 1, \dots, N-1$.

Estimating the error $\|f - f_N\|$ we obtain for $j = 0, \dots, M_k - 1$, $1 \leq k \leq N-1$

$$\begin{aligned}
\|f - f_N\|_{C(\Delta_{k,j}^1)} &\leq \frac{c(t_{k+1} - t_k)^s}{M_k^s s!} \left(\frac{N}{k} \right)^{v\gamma} \left(1 + \left| \ln^u \left(\frac{k}{N} \right)^v \right| \right) \leq \\
&\leq c \left(\left(\left(\frac{k+1}{N} \right)^v - \left(\frac{k}{N} \right)^v \right) \frac{1}{M_k} \right)^s \left(\frac{N}{k} \right)^{v\gamma} \left(1 + \left| \ln^u \left(\frac{N}{k} \right)^v \right| \right) \leq \frac{c}{N^s}.
\end{aligned}$$

One can derive similarly $\|f - f_N\|_{C(\Delta_{k,j}^2)} \leq c/N^s$, $k = 1, 2, \dots, N-1$.

For $k = 0$, the following estimate is true: $\|f - f_N\|_{C(\Delta_{0,0}^1)} \leq E_{s-1}(f, \Delta_{0,0}^1)(1 + \lambda_s)$.

Using the Taylor expansion $T_r(f, \Delta_{0,0}^1, -1)$ we have

$$\begin{aligned} E_{s-1}(f, \Delta_{0,0}^1) &\leq \|f - T_r(f, \Delta_{0,0}^1, -1)\|_{C(\Delta_{0,0}^1)} \leq \\ &\leq \frac{1}{r!} \max_{t \in \Delta_{0,0}^1} \int_{-1}^t \frac{(1 + |\ln^u(1 + \tau)|)}{(1 + \tau)^\mu} (t - \tau)^r d\tau \leq ch_{00}^{r+1-\mu} |\ln^u h_{00}| \leq cN^{-s}. \end{aligned}$$

Recall $h_{00} = N^{-v}/M_0 \leq cN^{-v}(\ln N)^{-u/(r+1-\mu)}$. Hence, $\|f - f_N\|_{C(\Delta_{0,0}^1)} \leq cN^{-s}$.

One can estimate the norms $\|f - f_N\|_{C(\Delta_{0,j}^1)}$, and $\|f - f_N\|_{C(\Delta_{0,j}^2)}$, $j = 0, 1, \dots, M_0 - 1$ in a similar way.

Thus, we have obtained

$$\|f - f_N\|_{C(\Omega)} \leq cN^{-s}. \quad (3.6)$$

It remains to estimate the number n of nodes of the local spline f_N . Repeating the arguments used in the proof of the preceding theorem we derive $n \asymp N$ since the specific value of q did not enter in this calculation. Thus, we have $\|f - f_N\|_{C(\Omega)} \leq cn^{-s}$ and $d_n(Q_{r\gamma}^r(\Omega, 1), C) \leq cn^{-s}$. Comparing the last inequality with the estimate $\delta_n(Q_{r\gamma}^r(\Omega, 1)) \geq cn^{-s}$ we complete the proof of Theorem. \square

4. Widths of the classes $\bar{Q}_{r\gamma}^u(\Omega, 1)$ and $Q_{r\gamma}^u(\Omega, 1)$, $\Omega = [-1, 1]^l$. In this section we estimate the Kolmogorov and Babenko widths for each of the functional classes $\bar{Q}_{r\gamma}^u(\Omega, 1)$ and $Q_{r\gamma}^u(\Omega, 1)$, $\Omega = [-1, 1]^l$, $l = 2, 3, \dots$.

THEOREM 4.1. Let $\Omega = [-1, 1]^l$, $l \geq 2$, $u = 1, 2, \dots$, $v = s/(s - \gamma)$. The following estimates hold

$$\delta_n(\bar{Q}_{r\gamma}^u(\Omega, 1)) \asymp d_n(\bar{Q}_{r\gamma}^u(\Omega, 1), C) \asymp n^{-s/l} \quad (4.1)$$

if $v < l/(l - 1)$,

$$cn^{-s/l}(\ln n)^{u-1+s/l} \leq \delta_n(\bar{Q}_{r\gamma}^u(\Omega, 1)) \leq 2d_n(\bar{Q}_{r\gamma}^u(\Omega, 1), C) \leq \quad (4.2)$$

$$\leq c \begin{cases} n^{-s/l}(\ln n)^{us/r}, & u/r \geq 1/l + (u - 1)/s, \\ n^{-s/l}(\ln n)^{u-1+s/l}, & u/r \leq 1/l + (u - 1)/s \end{cases} \quad (4.3)$$

if $v = l/(l - 1)$.

Proof. We start to estimate the Kolmogorov widths. First, we construct a local spline not necessarily continuous which approximates the functions of the classes $\bar{Q}_{r\gamma}^u(\Omega, 1)$ for $v \leq l/(l - 1)$ and has the error given in the right-hand side of (4.1) – (4.3). Afterwards we construct a continuous local spline having the same error of approximation. This requires some modifications which we indicate below.

Let Δ^k denote the set

$$\Delta^k = \left\{ t \in \Omega : \left(\frac{k}{N} \right)^v \leq d(t, \Gamma) \leq \left(\frac{k+1}{N} \right)^v, \quad k = 0, 1, \dots, N-1, \right\}$$

where $d(t, \Gamma)$ is as in Definition 2.1.

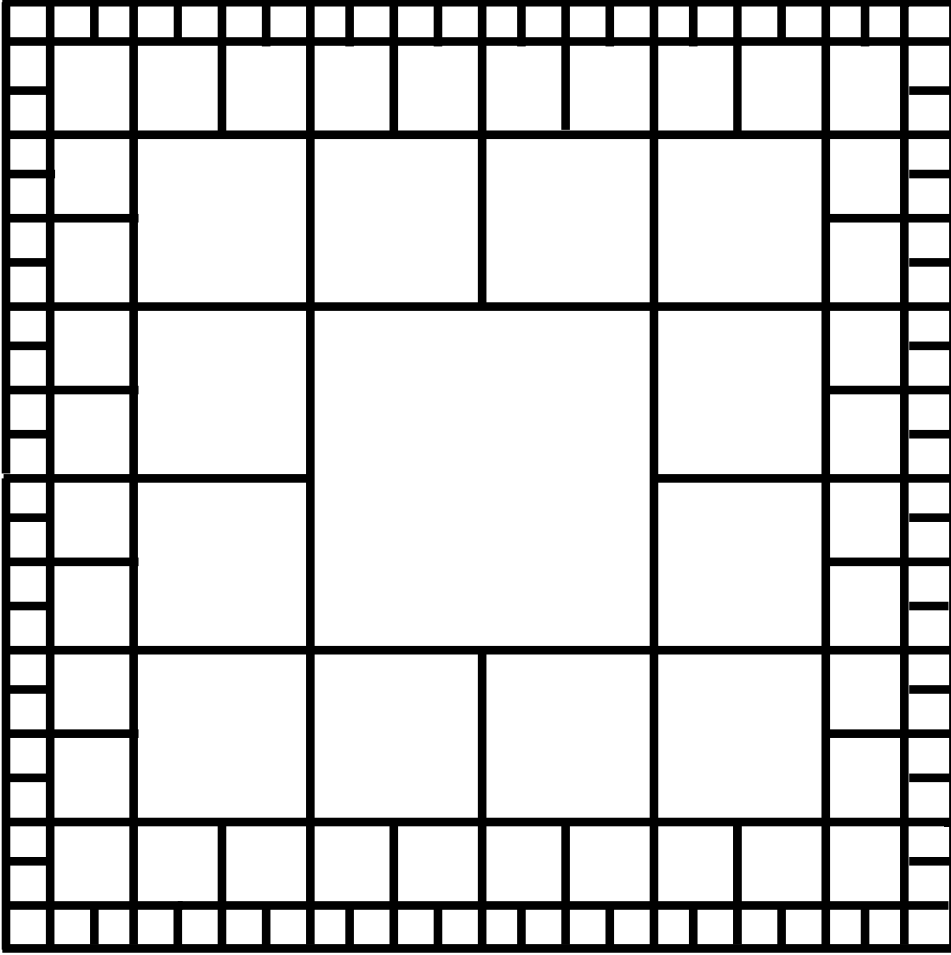


FIG. 4.1. Partition of the domain $\Omega = [-1, 1]^2$ by subdomains Δ_{i_1, i_2}^k .

We now partition the domains Δ^k , $k = 0, \dots, N-1$, in the following way. Decompose each Δ^k into cubes and parallelepipeds $\Delta_{i_1, \dots, i_l}^k$ with their edges parallel to the axes. The lengths of edges are not less than the value h_k and less than $2h_k$, $h_k = ((k+1)/N)^v - (k/N)^v$, $k = 0, 1, \dots, N-1$. (See Fig 4.1)

Let estimate the number of $\Delta_{i_1, \dots, i_l}^k$, $k = 0, 1, \dots, N-1$. Clearly,

$$\begin{aligned}
 n &\geq 1 + m \sum_{k=0}^{N-1} \left[\frac{2 - 2((k+1)/N)^v}{2h_k} \right]^{l-1} = 1 + m \sum_{k=0}^{N-1} \left[\frac{N^v - (k+1)^v}{(k+1)^v - k^v} \right]^{l-1} \geq \\
 &\geq c \begin{cases} N^{v(l-1)}, & v > l/(l-1), \\ N^l, & v < l/(l-1), \\ N^l \ln N, & v = l/(l-1), \end{cases} \quad (4.4)
 \end{aligned}$$

where m is the number of faces in Ω .

Similarly,

$$n \leq 1 + m \sum_{k=0}^{N-1} \left(\frac{2 - 2(k/N)^v}{h_k} + 1 \right)^{l-1} \leq c \begin{cases} N^{v(l-1)}, & v > l/(l-1), \\ N^l, & v < l/(l-1), \\ N^l \ln N, & v = l/(l-1), \end{cases} \quad (4.5)$$

Thus,

$$n \asymp \begin{cases} N^{v(l-1)}, & v > l/(l-1), \\ N^l, & v < l/(l-1), \\ N^l \ln N, & v = l/(l-1), \end{cases}$$

Let $M_0 = \lceil (\ln N)^{u/r} \rceil$, $M_k = \lceil (\ln N/k)^{(u-1)/s} \rceil$, $k = 1, 2, \dots, N-1$. Dividing each edge of $\Delta_{i_1, \dots, i_l}^k$ into M_k equal subintervals and passing the planes parallel to the coordinate planes through the points of division we partition $\Delta_{i_1, \dots, i_l}^k$ into $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k$.

In Section 3 we used the interpolating polynomial $P_s(f, [a, b])$ for functions f of one variable. As a next step we consider a possible multivariate counterpart. More precisely, for a function $f(t_1, \dots, t_l)$ of l variables on $[a_1, b_1; \dots; a_l, b_l]$ we define the interpolating polynomial $P_{s, \dots, s}(f, [a_1, b_1; \dots; a_l, b_l])$ iteratively: $P_{s, \dots, s}(f, [a_1, b_1; \dots; a_l, b_l]) = P_s^{t_1}(P_s^{t_2}(\dots P_s^{t_l}(f; [a_l, b_l]); [a_{l-1}, b_{l-1}]); \dots; [a_1, b_1])$. This polynomial then is of degree $s-1$ in each of the variables t_1, \dots, t_l . In other words, $P_s^{t_l}(f; [a_l, b_l])$ interpolates $f(t_1, \dots, t_l)$ with respect to $t_l \in [a_l, b_l]$; $P_s^{t_{l-1}}(P_s^{t_l}(f; [a_l, b_l]); [a_{l-1}, b_{l-1}])$ interpolates $P_s^{t_l}(f, [a_l, b_l])$ in $t_{l-1} \in [a_{l-1}, b_{l-1}]$ etc. The polynomial $P_{s, \dots, s}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k)$ interpolates $f(t_1, \dots, t_l)$ in each $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k$. We piece together the interpolating polynomials $P_{s, \dots, s}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k)$ and construct a local spline f_N . Next we estimate the approximation of $f \in \tilde{Q}_{r\gamma}^u(\Omega, 1)$ by f_N .

Let $k = 0$. Then, $\|f - P_{s, \dots, s}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0)\| \leq c E_{r-1, \dots, r-1}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0) \lambda_r^l$, where $E_{r, \dots, r}(f; \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0)$ is the best approximation to a function f in the space C by a polynomial of degree r in each variable in $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0$.

To estimate $E_{r, \dots, r}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0)$, we use Taylor's expansion with the remainder in the integral form (see e.g. (Nikolskii, 1975))

$$f(t_1, \dots, t_l) = \sum_{\alpha=0}^r \frac{1}{\alpha!} \sum_{j_1=1}^l \dots \sum_{j_\alpha=1}^l (t_{j_1} - t_{j_1}^0) \dots (t_{j_\alpha} - t_{j_\alpha}^0) \frac{\partial^\alpha f(t^0)}{\partial t_{j_1} \dots \partial t_{j_\alpha}} + R_{r+1}(t), \quad (4.6)$$

where

$$\begin{aligned} R_{r+1}(t) &= \frac{1}{r!} \int_0^1 (1-\tau)^r \sum_{j_1=1}^l \dots \sum_{j_{r+1}=1}^l (t_{j_1} - t_{j_1}^0) \dots (t_{j_{r+1}} - t_{j_{r+1}}^0) \frac{\partial^{r+1} f(t^0 + \tau(t - t^0))}{\partial t_{j_1} \dots \partial t_{j_{r+1}}} d\tau = \\ &= (r+1) \sum_{|\alpha|=r+1} \frac{(t - t^0)^\alpha}{\alpha!} \int_0^1 (1-\tau)^r f^{(\alpha)}(t^0 + \tau(t - t^0)) d\tau. \end{aligned}$$

With t, t_0 in the domain $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0$, which has a nonempty intersection with the boundary $\Gamma = \partial\Omega$, we trivially have $d(t^0 + \tau(t - t^0), \Gamma) \leq h_{00} = h_0 / \lceil (\ln N)^{u/r} \rceil$,

and thus $|f^{(r)}(t^0 + \tau(t - t^0))| \leq 1 + |ln^u h_{00}|$, which immediately yields

$$\begin{aligned} E_{r-1, \dots, r-1}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0) &\leq ch_{00}^r \int_0^1 (1 - \tau)^{r-1} (1 + |\ln^u(\tau h_{00})|) d\tau \\ &\leq ch_{00}^r \ln^u h_{00} \leq c \left(\frac{1}{N} \right)^s, \end{aligned}$$

where $h_{00} = h_0/M_0$, $h_0 = 1/N^v$.

Hence,

$$\|f - P_{s, \dots, s}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0)\|_{C(\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0)} \leq cN^{-s}. \quad (4.7)$$

The estimate is valid for all $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0$.

Now let $1 \leq k \leq N - 1$. Then,

$$\begin{aligned} &\|f - P_{s, \dots, s}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k)\|_{C(\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k)} \leq \\ &\leq c \left(\left(\left(\frac{k+1}{N} \right)^v - \left(\frac{k}{N} \right)^v \right) \frac{1}{(\ln \frac{N}{k})^{(u-1)/s}} \right)^s \frac{(1 + |\ln(\frac{k}{N})^v|)^{u-1}}{(\frac{k}{N})^{v\gamma}} \leq \frac{c}{N^s}. \end{aligned} \quad (4.8)$$

Combining (4.7) and (4.8) we conclude

$$\|f - f_N\| \leq cN^{-s}. \quad (4.9)$$

Estimating the number of nodes used in constructing f_N we study two cases i) $v < l/(l-1)$ and ii) $v = l/(l-1)$.

i). Let $v < l/(l-1)$. The upper estimate follows immediately from the chain of inequalities

$$\begin{aligned} n &\leq m \sum_{k=1}^{N-1} \left(\frac{2 - 2(\frac{k}{N})^v}{(\frac{k+1}{N})^v - (\frac{k}{N})^v} \right)^{l-1} M_k^l + mN^{v(l-1)} [\ln N]^{lu/r} \leq \\ &\leq cN^{v(l-1)} (\ln N)^{lu/r} + c \sum_{k=1}^{N-1} \left(\frac{2N^v - 2k^v}{v(k+\theta)^{v-1}} \right)^{l-1} \left(1 + \left(\ln \frac{N}{k} \right)^{\frac{u-1}{s}} \right)^l \leq \\ &\leq cN^{v(l-1)} (\ln N)^{lu/r} + c \sum_{k=1}^{N-1} \frac{N^{v(l-1)}}{k^{(v-1)(l-1)}} \left(\left(\ln \frac{N}{k} \right)^{\frac{(u-1)l}{s}} + 1 \right) \leq cN^l, \end{aligned} \quad (4.10)$$

where m is the number of faces of Ω .

The inequalities (4.9) and (4.10) yield $\|f - f_N^*\| \leq cn^{-s/l}$ for $v < l/(l-1)$.

ii). Let $v = l/(l-1)$. Just as for $v < l/(l-1)$, the upper bound follows from the chain of inequalities

$$n \leq m \sum_{k=1}^{N-1} \left(\frac{2 - 2(\frac{k}{N})^v}{(\frac{k+1}{N})^v - (\frac{k}{N})^v} \right)^{l-1} M_k^l + mN^{v(l-1)} [\ln N]^{lu/r} \leq$$

$$\begin{aligned}
&\leq cN^l(\ln N)^{lu/r} + c \sum_{k=1}^{N-1} \left(\frac{N^v}{k^{v-1}} \right)^{l-1} \left[\ln \frac{N}{k} \right]^{(u-1)l/s} \leq \\
&\leq cN^l(\ln N)^{lu/r} + \frac{cN^l}{N^{(v-1)(l-1)}} \sum_{k=1}^{N-1} \frac{N^{(v-1)(l-1)}}{k^{(v-1)(l-1)}} \left(\ln \frac{N}{k} \right)^{(u-1)l/s} \leq \\
&\leq cN^l(\ln N)^{lu/r} + cN^{l-1} \int_1^N \frac{N}{x} \left(\ln \frac{N}{x} \right)^{(u-1)l/s} dx \leq \\
&\leq c \begin{cases} N^l(\ln N)^{lu/r}, & lu/r \geq 1 + (u-1)l/s, \\ N^l(\ln N)^{(u-1)l/s+1}, & lu/r \leq 1 + (u-1)l/s. \end{cases} \quad (4.11)
\end{aligned}$$

From (4.9), (4.11), and for $v = l/(l-1)$, it follows

$$\|f - f_N\| \leq c \begin{cases} \left(\frac{1}{n}\right)^{s/l} (\ln n)^{us/r}, & u/r \geq 1/l + (u-1)/s, \\ \left(\frac{1}{n}\right)^{s/l} (\ln n)^{u-1+s/l}, & u/r \leq 1/l + (u-1)/s. \end{cases}$$

where n is the number of nodes of the local spline.

Now we describe the necessary modifications to the computations above in order to construct a continuous local spline approximating $\bar{Q}_{r\gamma}^u(\Omega, 1)$ for $v \leq l/(l-1)$ and having the estimate given in (4.1) - (4.3). Let Δ^k be defined as above and $h_k = ((k+1)/N)^v - (k/N)^v$, $h_k^* = h_k/M_k$, $k = 0, 1, \dots, N-1$. We then shall use the following modified partitioning of Ω . As before decompose Δ^{N-2} into cubes or parallelepipeds $\Delta_{i_1, \dots, i_l}^{N-2}$ with edges parallel to the axes, and whose lengths is not less than h_{N-2} and does not exceed then $2h_{N-2}$, but now choose the partition in such a way that the vertices of Δ^{N-1} are contained in the set consisting of all vertices of the subdomains $\Delta_{i_1, \dots, i_l}^{N-2}$,

Next, partition each $\Delta_{i_1, \dots, i_l}^{N-2}$ into cubes or parallelepipeds $\Delta_{i_1, \dots, i_l; j_1, \dots, j_r}^{N-2}$ with edges parallel to the axes. We divide each edge of the subdomain $\Delta_{i_1, \dots, i_l}^{N-2}$ into M_{N-2} equal subintervals and pass the planes parallel to the coordinate planes through the points of division. Finally we decomposed Δ^{N-2} into $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^{N-2}$.

To decompose Δ^{N-3} into cubes and parallelepipeds $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^{N-3}$ we pass the planes parallel to the coordinate planes through the vertices of $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^{N-2}$, which are located on a common face of the hyperplane $\Delta^{N-2} \cap \Delta^{N-3}$. Denote the obtained subdomains by $g_{i_1, \dots, i_l; j_1, \dots, j_l}^{N-3}$. Let $h_{N-3}^* = h_{N-3}/M_{N-3}$. Consider $g_{i_1, \dots, i_l; j_1, \dots, j_l}^{N-3} = [a_1, b_1; \dots; a_l, b_l]$. If the length of the edge (a_k, b_k) exceeds $2h_{N-3}^*$, we divide (a_k, b_k) into $[|b_k - a_k|/h_{N-3}^*]$ equal subintervals and pass the planes parallel to the coordinate planes through the points of division. We shall refer to the result of this procedure as $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^{N-3}$. This way we have Δ^{N-3} decomposed into $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^{N-3}$. Continuing this process we partition the domain Ω into subdomains $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k$, $k = 0, 1, \dots, N-2$.

One estimates the total number of $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k$ in Ω using (4.10), (4.11).

Now we construct the continuous spline f_N approximating a function f of l variables. The polynomial $P_{s, \dots, s}(f, \Delta^{N-1})$ interpolates f in Δ^{N-1} , $P_{s, \dots, s}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^{N-2})$

interpolates \tilde{f} in $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^{N-2}$. We say that the function \tilde{f} equals to f at all points of interpolation except for those located on the hypersurface $\Delta^{N-1} \cap \Delta^{N-2}$. At those points, \tilde{f} equals to $P_{s, \dots, s}(f, \Delta^{N-1})$. Continuing this process, we construct the interpolating polynomials $P_{s, \dots, s}(\tilde{f}, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k)$, $k = N-3, \dots, 1, 0$.

Next we piece together all the interpolating polynomials $P_{s, \dots, s}(f, \Delta^{N-1})$, $P_{s, \dots, s}(\tilde{f}, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k)$, $k = 0, 1, 2, \dots, N-2$, that interpolate f within each Δ^{N-1} , $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k$, and construct the continuous local spline f_N^* .

Repeating the above computations for a non-continuous local spline we obtain $\|f - f_N^*\|_{C(\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k)} \leq cN^{-s}$, $k = 0, 1, 2, \dots, N-2$, $\|f - f_N^*\|_{C(\Delta^{N-1})} \leq cN^{-s}$.

Therefore

$$\|f - f_N^*\|_{C(\Omega)} \leq cN^{-s}. \quad (4.12)$$

Using the inequalities (4.10), (4.11), (4.12) we have proved the following statements

$$d_n(\bar{Q}_{r\gamma}^u(\Omega, 1), C) \leq cn^{-s/l} \quad (4.13)$$

if $v < l/(l-1)$,

$$d_n(\bar{Q}_{r\gamma}^u(\Omega, 1), C) \leq c \begin{cases} n^{-s/l}(\ln n)^{us/r}, & u/r \geq 1/l + (u-1)/s, \\ n^{-s/l}(\ln n)^{u-1+s/l}, & u/r \leq 1/l + (u-1)/s \end{cases} \quad (4.14)$$

if $v = l/(l-1)$.

Let estimate $\delta_n(\bar{Q}_{r\gamma}^u(\Omega, 1))$ for $v = s/(s-\gamma)$, $v < l/(l-1)$.

Note that for every given positive integer u we have $Q_{r\gamma}(\Omega, 1) \subset \bar{Q}_{r\gamma}^u(\Omega, 1)$, which together with Theorem 2.13 yields

$$\delta_n(\bar{Q}_{r\gamma}^u(\Omega, 1)) \geq \delta_n(Q_{r\gamma}(\Omega, 1)) \asymp n^{-s/l}, v < l/(l-1). \quad (4.15)$$

Let estimate $\delta_n(\bar{Q}_{r\gamma}^u(\Omega, 1))$ for $v = s/(s-\gamma)$, $v = l/(l-1)$.

We decompose the domain Ω into subdomains $\Delta_{i_1, \dots, i_l}^k$, $k = 0, 1, \dots, N-1$ following the procedure which was described above in the proof of Theorem (see the part of constructing a not necessarily continuous local spline).

Let

$$M_k = \begin{cases} \lceil (\ln N)^{(u-1)/s} \rceil, & k = 0 \\ \lceil (\ln \frac{N}{k})^{(u-1)/s} \rceil, & k = 1, 2, \dots, N-1. \end{cases}$$

We divide each edge of $\Delta_{i_1, \dots, i_l}^k$, $k = 0, 1, \dots, N-1$ into M_k equal subintervals and pass the planes parallel to coordinate planes through the points of division. This way we have $\Delta_{i_1, \dots, i_l}^k$, $k = 0, 1, \dots, N-1$ decomposed into $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k$, $k = 0, 1, \dots, N-1$.

Let estimate the number $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k$, $k = 0, 1, \dots, N-1$. Clearly

$$n \asymp m \sum_{k=1}^{N-1} \left(\frac{2 - 2(\frac{k}{N})^v}{(\frac{k+1}{N})^v - (\frac{k}{N})^v} \right)^{l-1} M_k^l + mN^{v(l-1)}(\ln N)^{l(u-1)/s} \asymp$$

$$\asymp N^l (\ln N)^{l(u-1)/s} + N^{l-1} \int_1^N \frac{N}{x} \left(\ln \frac{N}{x} \right)^{(u-1)l/s} dx \asymp N^l (\ln N)^{(u-1)l/s+1}.$$

Let $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k = [b_{i_1, j_1}, b_{i_1, j_1+1}; \dots, b_{i_l, j_l}, b_{i_l, j_l+1}]$. Introduce the functions

$$\begin{aligned} & \varphi_{i_1, \dots, i_l; j_1, \dots, j_l}^k(t) = \\ & = \begin{cases} A_k \frac{((t_1 - b_{i_1, j_1})(b_{i_1, j_1+1} - t_1) \dots (t_l - b_{i_l, j_l})(b_{i_l, j_l+1} - t_l))^s}{(h_k/M_k)^{(2l-1)s}((k+1)/N)^{v\gamma}} \left(1 + \left| \ln^{u-1} \left(\frac{k+1}{N} \right)^v \right| \right), \\ t \in \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k, 0, \quad t \in \Omega \setminus \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k, \end{cases} \quad (4.16) \end{aligned}$$

$k = 0, 1, \dots, N-2$.

Let $\Delta^{N-1} = [b_{i_1}, b_{i_1+1}; \dots, b_{i_l}, b_{i_l+1}]$. Introduce the function

$$\varphi^{N-1}(t) = \begin{cases} A_{N-1} \frac{((t_1 - b_{i_1})(b_{i_1+1} - t_1) \dots (t_l - b_{i_l})(b_{i_l+1} - t_l))^s}{h_{N-1}^{(2l-1)s}}, & t \in \Delta^{N-1}, \\ 0, & t \in \Omega \setminus \Delta^{N-1}, \end{cases} \quad (4.17)$$

Constants A_k , $k = 0, 1, \dots, N-1$, are chosen such that

$$|D^s \varphi_{i_1, \dots, i_l}^k| \leq \left(1 + \left| \ln^{u-1} \left(\frac{k+1}{N} \right)^v \right| \right) / \left(\frac{k+1}{N} \right)^{v(s-r)}.$$

Constant A_{N-1} is chosen such that $|D^s \varphi^{N-1}| \leq 1$.

Let $\xi(t)$ be a linear combination $\xi(t) = \sum_{k, i_1, \dots, i_l; j_1, \dots, j_l} C_{i_1, \dots, i_l; j_1, \dots, j_l}^k \varphi_{i_1, \dots, i_l; j_1, \dots, j_l}^k(t)$,

where $|C_{i_1, \dots, i_l; j_1, \dots, j_l}^k| \leq 1$. Here the summation is taken over all domains $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k$ of Ω .

Repeating the arguments presented in Anuchina et al. (1979); Babenko (1985); Boykov (1998) we have $\delta_n(\bar{Q}_{r\gamma}^u(\Omega, 1)) \geq cN^{-s} \geq cn^{-s/l}(\ln n)^{u-1+s/l}$.

From the inequalities (4.13) – (4.14) and estimates of the Babenko widths we complete the proof of Theorem.

□

Remark. Let $v = l/(l-1)$. The estimate $d_n(\bar{Q}_{r\gamma}^u(\Omega, 1), C) \geq cn^{-s/l}(\ln n)^{u-1+s/l}$ follows from the definition of $\xi(t)$ and Lemma 2.8.

For $v > l/(l-1)$, we state the following

THEOREM 4.2. Let $\Omega = [-1, 1]^l$, $l \geq 2$, $u = 1, 2, \dots$, $v = s/(s-\gamma)$, $v > l/(l-1)$. The estimate $\delta_n(\bar{Q}_{r\gamma}^u(\Omega, 1)) \geq cn^{-(s-\gamma)/(l-1)} \ln^{u-1} N$ holds.

Proof. Let Δ^0 be the set $\Delta^0 = \{t \in \Omega : 0 \leq d(t, \Gamma) \leq (1/N^v) = \rho_0\}$.

Let Δ^k be the set $\Delta^k = \{t \in \Omega : \rho_{k-1} \leq d(t, \Gamma) \leq \rho_k \leq 1\}$, where ρ_k is defined by $h_k^s/\rho_k^\gamma = N^{-s} \ln^{u-1} N$, and $h_k = \rho_k - \rho_{k-1}$, $k = 1, 2, \dots, m$. Here m is the largest integer value when $\rho_m \leq 1$. If $\rho_m = 1$, then Ω is decomposed into Δ^k , $k = 0, 1, \dots, m$. If $\rho_m < 1$, then Δ^{m+1} is the set $\Delta^{m+1} = \{t \in \Omega : \rho_m \leq d(t, \Gamma) \leq 1\}$.

Without loss of generality we demonstrate our computations for $\rho_m = 1$. Now we show that the equations $h_k^s/\rho_k^\gamma = N^{-s} \ln^{u-1} N$ are solvable.

Let $\rho_k^* = (k/N)^v$, $k = 0, 1, \dots, N$, $h_k^* = \rho_k^* - \rho_{k-1}^*$, $k = 1, \dots, N$. Then $h_1^{*s}/\rho_1^{*\gamma} = 1/N^s$ if $k = 1$. For $k = 2, \dots, N$, we have

$$\frac{h_k^{*s}}{\rho_k^{*\gamma}} = \frac{(k^v - (k-1)^v)^s}{(k/N)^{v\gamma} N^{vs}} = \frac{(v(k-\Theta)^{v-1})^s}{k^{v\gamma}} \frac{1}{N^s} \geq \left(\frac{k-1}{k} \right)^{v\gamma} v^s \frac{1}{N^s} \geq \left(\frac{1}{2} \right)^{v\gamma} v^s \frac{1}{N^s}.$$

Thus there exists a sequence $\rho_k^* = (k/N)^v$, $k = 0, 1, \dots, N$, such that $h_k^{*s}/\rho_k^{*\gamma} \geq (1/2)^{v\gamma}(v/N)^s = cN^{-s}$, $h_k^* = \rho_k^* - \rho_{k-1}^*$.

On the other hand, $\varphi(\rho) = (\rho - \rho_{k-1})^s/\rho^\gamma$ is an increasing function if $\rho > \rho_{k-1}$ for any ρ_{k-1} .

Therefore there exists a sequence ρ_k such that $(\rho_k - \rho_{k-1})^s/(\rho_k)^\gamma \geq N^{-s} \ln^{u-1} N$, moreover $h_k = \rho_k - \rho_{k-1} > h_k^* = \rho_k^* - \rho_{k-1}^*$, $k = 1, \dots, m$.

Hence the number m of Δ^k , $k = 0, 1, \dots, m$ is less than N . We decompose each Δ^k into cubes or parallelepipeds $\Delta_{i_1, \dots, i_l}^k$ in a way described above in the proof of the Theorem 4.1 (see the part of constructing a not necessarily continuous local spline). Clearly, the total number of $\Delta_{i_1, \dots, i_l}^k$, $k = 0, 1, \dots, m$ is equal to $n \asymp n_0 \asymp N^{v(l-1)}$, where n_0 is the number of $\Delta_{i_1, \dots, i_l}^0$.

Let $\Delta_{i_1, \dots, i_l}^k = [b_{i_1}^k, b_{i_1+1}^k; \dots; b_{i_l}^k, b_{i_l+1}^k]$, $k = 0, 1, \dots, m$. Introduce the functions

$$\varphi_{i_1, \dots, i_l}^0(t_1, \dots, t_l) = \begin{cases} A_0 \frac{((t_1 - b_{i_1}^0)(b_{i_1+1}^0 - t_1) \dots (t_l - b_{i_l}^0)(b_{i_l+1}^0 - t_l))^s}{h_0^{(2l-1)s}} N^{v\gamma} \ln^{u-1} N, \\ t \in \Delta_{i_1, \dots, i_l}^0, 0, t \in \Omega \setminus \Delta_{i_1, \dots, i_l}^0; \end{cases}$$

$$\varphi_{i_1, \dots, i_l}^k(t_1, \dots, t_l) = \begin{cases} A_k \frac{((t_1 - b_{i_1}^k)(b_{i_1+1}^k - t_1) \dots (t_l - b_{i_l}^k)(b_{i_l+1}^k - t_l))^s}{h_k^{(2l-1)s} \rho_k^\gamma}, \\ t \in \Delta_{i_1, \dots, i_l}^k, 0, t \in \Omega \setminus \Delta_{i_1, \dots, i_l}^k, \end{cases}$$

$k = 1, 2, \dots, m$. Constants A_k , $k = 0, 1, \dots, m$, are chosen such that $|D^s \varphi_{i_1, \dots, i_l}^0| \leq N^{v\gamma} \ln^{u-1} N$, $|D^s \varphi_{i_1, \dots, i_l}^k| \leq 1/\rho_k^\gamma$. Obviously, such constants exist and do not depend on N, u, γ .

Let estimate the maximum values of $\varphi_{i_1, \dots, i_l}^k(t)$, $k = 0, 1, \dots, m$. Clearly

$$\varphi_{i_1, \dots, i_l}^0(t) \geq ch_0^s N^{v\gamma} \ln^{u-1} N = cN^{-v(s-\gamma)} \ln^{u-1} N = cN^{-s} \ln^{u-1} N,$$

$$\varphi_{i_1, \dots, i_l}^k(t) \geq ch_k^s/\rho_k^\gamma = cN^{-s} \ln^{u-1} N, h_k = \rho_k - \rho_{k-1}, \quad k = 0, 1, \dots, m.$$

Let $\xi(t)$ be a linear combination $\xi(t) = \sum_{k, i_1, \dots, i_l} C_{i_1, \dots, i_l}^k \varphi_{i_1, \dots, i_l}^k(t)$, where $|C_{i_1, \dots, i_l}^k| \leq$

1. Here the summation is taken over all domains $\Delta_{i_1, \dots, i_l}^k$ of Ω .

Repeating the arguments presented in Anuchina et al. (1979); Babenko (1985); Boykov (1998), we have $\delta_n(\bar{Q}_{r\gamma}^u(\Omega, 1)) \geq cN^{-s} \ln^{u-1} N = cn^{-(s-\gamma)/(l-1)} \ln^{u-1} n$. \square

Remark. The estimate $d_n(\bar{Q}_{r\gamma}^u(\Omega, 1), C) \geq cn^{-(s-\gamma)/(l-1)} \ln^{u-1} n$ follows from the definition of $\xi(t)$ and Lemma 2.8.

Let $v > l/(l-1)$. First, we construct a local spline not necessary continuous which approximates the functions of $\bar{Q}_{r\gamma}^u(\Omega, 1)$ for $v > l/(l-1)$ and has the error not exceeding $c(\ln^u n)n^{-(s-\gamma)/(l-1)}$. Afterwards we construct a continuous local spline having the same error of approximation.

When constructing a local spline we employ the same process as used in the proof of Theorem 4.1 (see the part of constructing a not necessarily continuous local spline). We define the domains Δ^k and partition them into $\Delta_{i_1, \dots, i_l}^k$, $k = 0, 1, \dots, N-2$, in a similar way we did for $v \leq l/(l-1)$.

Clearly, the number n of $\Delta_{i_1, \dots, i_l}^k$ is estimated by

$$n \asymp N^{v(l-1)}. \quad (4.18)$$

The polynomial $P_{s, \dots, s}(f; \Delta_{i_1, \dots, i_l}^k)$ interpolates f in $\Delta_{i_1, \dots, i_l}^k$, $k = 0, 1, \dots, N-1$. Hence the local spline f_N is composed of the polynomials $P_{s, \dots, s}(f; \Delta_{i_1, \dots, i_l}^k)$, $k = 0, 1, \dots, N-1$.

It is easy to see that for $1 \leq k \leq N-1$ the following estimate holds

$$\|f - f_N\|_{C(\Delta_{i_1, \dots, i_l}^k)} \leq cN^{-s}(\ln N)^{u-1}. \quad (4.19)$$

Indeed,

$$\|f - f_N\|_{C(\Delta_{i_1, \dots, i_l}^k)} \leq ch_k^s \frac{|\ln(\frac{k}{N})^v|^{u-1}}{((k/N)^v)^\gamma} \leq ch_k^s \left(\frac{N}{k}\right)^{v\gamma} (\ln N)^{u-1} = \frac{c}{N^s} (\ln N)^{u-1}.$$

Let $k=0$. Without loss of generality we demonstrate our computations in $\Delta_{0, \dots, 0}^0 = [-1, t_1; -1, t_1; \dots; -1, t_1]$, where $t_1 = -1 + (\frac{1}{N})^v$. Using Taylor's expansion (4.6) we obtain

$$\begin{aligned} \|f - f_N\|_{C(\Delta_{0, \dots, 0}^0)} &\leq c\lambda_s^l E_{r-1, \dots, r-1}(f, \Delta_{0, \dots, 0}^0) \leq \\ &\leq c \max_{t \in \Delta_{0, \dots, 0}^0} \left| \sum_{|k|=r} \frac{1}{k!} \int_0^1 (1-\tau)^{r-1} (t-t^0)^k (1 + |\ln^u d(-1 + \tau(t_k + 1)), \Gamma|) d\tau \right| \leq \\ &\leq c \max_{t \in \Delta_{0, \dots, 0}^0} \left| \sum_{|k|=r} \frac{1}{k!} \int_0^1 (1-\tau)^{r-1} (t-t^0)^k |\ln^u \tau(t_k + 1)| d\tau \right| \leq ch_0^r |\ln^u h_0| \leq c \frac{\ln^u N}{N^s}, \end{aligned}$$

where $t^0 = (-1, \dots, 1)$.

From the previous estimate and the equalities (4.18) we have $\|f - f_N\|_{C(\Omega)} \leq cN^{-s} \ln^u N \leq cn^{-(s-\gamma)/(l-1)} \ln^u n$.

To construct the continuous local spline f_N^* approximating $\bar{Q}_{r\gamma}^u(\Omega, 1)$ for $v > l/(l-1)$ and having the error $c(\ln^u n)n^{-(s-\gamma)/(l-1)}$, we employ all above constructions for the continuous local spline f_N^* approximating $\bar{Q}_{r\gamma}^u(\Omega, 1)$ when $v \leq l/(l-1)$, cf. the proof of 4.1.

Thus, for $0 \leq k \leq N-1$, the following estimates hold

$$\|f - f_N^*\|_{C(\Delta_{i_1, \dots, i_l}^k)} \leq cN^{-s}(\ln N)^{u-1}, k = 1, \dots, N-1, \quad (4.20)$$

$$\|f - f_N^*\|_{C(\Delta_{0, \dots, 0}^0)} \leq ch_0^r |\ln^u h_0| \leq cN^{-s} \ln^u N. \quad (4.21)$$

From the previous estimates and the equality (4.18) we have $\|f - f_N^*\|_{C(\Omega)} \leq cN^{-s} \ln^u N \leq cn^{-(s-\gamma)/(l-1)} \ln^u n$.

Since the number of the nodes used in construction of the continuous local spline f_N^* is s^l in each $\Delta_{i_1, \dots, i_l}^k$, $k = 0, 1, \dots, N-1$, we state the following

THEOREM 4.3. Let $\Omega = [-1, 1]^l$, $l \geq 2$, $u = 1, 2, \dots$, $v = s/(s-\gamma)$, $v > l/(l-1)$. Then the estimate $d_n(\bar{Q}_{r\gamma}^u(\Omega, 1)) \leq cn^{-(s-\gamma)/(l-1)} \ln^u n$ holds.

To estimate the Kolmogorov widths $d_n(Q_{r\gamma}^u(\Omega, 1), C)$ for $u = 1, 2, \dots$, we use Definition 2.5 for $Q_{r\gamma}^u(\Omega, 1)$ and note that $\gamma = s - r - 1 + \mu$, $\mu = 1 + \gamma - [\gamma]$.

THEOREM 4.4. Let $\Omega = [-1, 1]^l$, $l \geq 2$, $u = 1, 2, \dots$, $v = s/(s-\gamma)$. Then

$$d_n(Q_{r\gamma}^u(\Omega, 1), C) \leq cn^{-s/l} \quad (4.22)$$

if $v < l/(l-1)$.

$$d_n(Q_{r\gamma}^u(\Omega, 1), C) \leq cn^{-s/l}(\ln n)^{us/(r+1-\mu)} \quad (4.23)$$

for $lu/(r+1-\mu) \geq ul/s+1$,

$$d_n(Q_{r\gamma}^u(\Omega, 1), C) \leq cn^{-s/l}(\ln n)^{(ul+s)/l} \quad (4.24)$$

for $lu/(r+1-\mu) < ul/s+1$, if $v = l/(l-1)$.

Proof. The proof of the theorem is similar to the proof of Theorem 4.1. First, we construct a local not necessarily continuous spline which approximates the functions of the class $Q_{r\gamma}^u(\Omega, 1)$ and has the error given in the right-hand sides of (4.22) – (4.24). Afterwards we construct a continuous local spline having the same error of approximation.

We decompose the domain Ω into subdomains $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k$, $k = 0, 1, \dots, N-1$ following the procedure which was described more than once in this section. (For instance, see the proof of Theorem 4.1). In doing so, we divide each edge of $\Delta_{i_1, \dots, i_l}^k$ into M_k equal subintervals, $M_k = \lceil (\ln N)^{u/(r+1-\mu)} \rceil$, $k = 0$, $M_k = \lceil (\ln(N/k))^{u/s} \rceil$, $k = 1, 2, \dots, N-1$ and pass the planes parallel to the coordinate planes through the points of division. To interpolate f in each of the obtained cubes or parallelepipeds $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k$ we use the polynomial $P_{s, \dots, s}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k)$ described in this section. Hence the local spline f_N is composed of the polynomials $P_{s, \dots, s}(f; \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k)$, $k = 0, 1, \dots, N-1$.

Remark. We will use polynomials $P_{s, \dots, s}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k)$ when $s \geq r+2$, and polynomials $P_{s+1, \dots, s+1}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k)$ when $s = r+1$. Without loss of generality we demonstrate our computations when $s \geq r+2$.

Estimating an approximation f_N to $f \in Q_{r\gamma}^u(\Omega, 1)$ we obtain for $1 \leq k \leq N-1$

$$\begin{aligned} & \|f - P_{s, \dots, s}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k)\|_{C(\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k)} \leq \\ & \leq c \left(\left(\left(\frac{k+1}{N} \right)^v - \left(\frac{k}{N} \right)^v \right) \frac{1}{(\ln \frac{N}{k})^{u/s}} \right)^s \frac{(1 + |\ln(\frac{k}{N})^v|)^u}{(\frac{k}{N})^{v\gamma}} \leq c \frac{1}{N^s}. \end{aligned} \quad (4.25)$$

If $k = 0$, then $\|f - P_{s, \dots, s}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0)\|_{C(\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0)} \leq c E_{r+1, \dots, r+1}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0) \lambda_s^l$.

Using Taylor's expansion with the remainder in integral form we have

$$\begin{aligned} E_{r+1, \dots, r+1}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0) & \leq ch_{00}^{r+1-\mu} \int_0^1 (1-\tau)^{r-1} (1 + |\ln^u(\tau h_{00})|) d\tau \leq \\ & \leq ch_{00}^{r+1-\mu} \ln^u h_{00} \leq cN^{-v(r+1-\mu)} = cN^{-s}, \end{aligned}$$

where $h_{00} = h_0/M_0$, $h_0 = (1/N)^v$.

Hence,

$$\|f - P_{s, \dots, s}(f, \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0)\|_{C(\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^0)} \leq cN^{-s}. \quad (4.26)$$

Combining (4.25) – (4.26) gives

$$\|f - f_N\| \leq cN^{-s}. \quad (4.27)$$

Now we estimate the number of nodes used in constructing f_N . As in the proof of Theorem 4.1, we study two cases i) $v < l/(l-1)$ and ii) $v = l/(l-1)$.

i). Let $v < l/(l-1)$. The estimate follows immediately from the chain of inequalities

$$\begin{aligned} n &\leq m \sum_{k=1}^{N-1} \left(\frac{2 - 2(\frac{k}{N})^v}{(\frac{k+1}{N})^v - (\frac{k}{N})^v} \right)^{l-1} M_k^l + 2mN^{v(l-1)} [\ln N]^{lu/(r+1-\mu)} \leq \\ &\leq cN^{v(l-1)} (\ln N)^{lu/(r+1-\mu)} + c \sum_{k=1}^{N-1} \left(\frac{2N^v - 2k^v}{v(k+\theta)^{v-1}} \right)^{l-1} \left(1 + \left(\ln \frac{N}{k} \right)^{\frac{u}{s}} \right)^l \leq \\ &\leq cN^{v(l-1)} (\ln N)^{lu/(r+1-\mu)} + c \sum_{k=1}^{N-1} \frac{N^{v(l-1)}}{k^{(v-1)(l-1)}} \left(\left(\ln \frac{N}{k} \right)^{\frac{ul}{s}} + 1 \right) \leq cN^l, \end{aligned}$$

where m is the number of faces of $\Delta_{i_1, \dots, i_l}^k$.

One can obtain that $n \geq cN^l$ is true in a similar way. Therefore,

$$n = cN^l. \quad (4.28)$$

The inequalities (4.27) – (4.28) yield $\|f - f_N\| \leq cn^{-s/l}$, where n is the number of nodes of the local spline.

ii). Let $v = l/(l-1)$. As we have already derived for $v < l/(l-1)$, the upper bound follows immediately from the chain of inequalities

$$\begin{aligned} n &\leq m \sum_{k=1}^{N-1} \left(\frac{2 - 2(\frac{k}{N})^v}{(\frac{k+1}{N})^v - (\frac{k}{N})^v} \right)^{l-1} M_k^l + 2mN^{v(l-1)} [\ln N]^{lu/(r+1-\mu)} \leq \\ &\leq cN^l (\ln N)^{lu/(r+1-\mu)} + cN^l (\ln N)^{(ul/s)+1}. \end{aligned}$$

It remains to express N in terms of n . It is necessary to study two cases:

i). the estimate $N \leq n^{1/l}/(\ln n)^{u/(r+1-\mu)}$ holds, if $lu/(r+1-\mu) \geq ul/s + 1$;

ii). $N \leq n^{1/l}/(\ln n)^{(ul+s)/(sl)}$ holds, if $lu/(r+1-\mu) < ul/s + 1$.

Thus, for $lu/(r+1-\mu) \geq ul/s + 1$

$$\|f - f_N\| \leq cn^{-s/l} (\ln n)^{us/(r+1-\mu)}; \quad (4.29)$$

for $lu/(r+1-\mu) < ul/s + 1$

$$\|f - f_N\| \leq cn^{-s/l} (\ln n)^{(ul+s)/l}. \quad (4.30)$$

To obtain the upper estimate of the Kolmogorov widths of the functional class $Q_{r\gamma}^u(\Omega, 1)$, we construct a continuous local spline which has the error given in the right-hand sides of (4.29), (4.30). For this purpose, we repeat the construction of the continuous local spline provided in the proof of Theorem 4.1. One can show that

the continuous local spline that approximates the functions of $Q_{r\gamma}^u(\Omega, 1)$ has the error given in the right-hand sides of (4.29), (4.30). \square

THEOREM 4.5. Let $\Omega = [-1, 1]^l$, $l \geq 2$, $u = 1, 2, \dots$, $v = s/(s - \gamma)$, $v < l/(l - 1)$. The estimate $\delta_n(Q_{r\gamma}^u(\Omega, 1)) \geq cn^{-s/l}$ holds.

Proof. It is easy to see that $Q_{r\gamma}(\Omega, 1) \subset Q_{r\gamma}^u(\Omega, 1)$. From Theorem 2.13 the estimate $\delta_n(Q_{r\gamma}(\Omega, 1)) \geq cn^{-s/l}$ follows. Therefore $\delta_n(Q_{r\gamma}^u(\Omega, 1)) \geq cn^{-s/l}$. \square

THEOREM 4.6. Let $\Omega = [-1, 1]^l$, $l \geq 2$, $u = 1, 2, \dots$, $v = s/(s - \gamma)$, $v = l/(l - 1)$. Then $\delta_n(Q_{r\gamma}^u(\Omega, 1)) \geq cn^{-s/l}(\ln n)^{u+s/l}$.

Proof. We decompose the domain Ω into subdomains $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k$, $k = 0, 1, \dots, N - 1$ following the procedure which was described in the proof of the Theorem 4.1 (see the part of Babenko widths estimates for $v = l/(l - 1)$).

Now, we introduce $M_k = \lceil (\ln N)^{u/s} \rceil$, $k = 0$; $M_k = \lceil (\ln(N/k))^{u/s} \rceil$, $k = 1, 2, \dots, N - 1$.

Let estimate the number $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k$, $k = 0, 1, \dots, N - 1$. Clearly

$$\begin{aligned} n &\asymp m \sum_{k=1}^{N-1} \left(\frac{2 - 2 \left(\frac{k}{N} \right)^v}{\left(\frac{k+1}{N} \right)^v - \left(\frac{k}{N} \right)^v} \right)^{l-1} M_k^l + 2mN^{v(l-1)}(\ln N)^{lu/s} \asymp \\ &\asymp N^l(\ln N)^{lu/s} + N^{l-1} \int_1^N \frac{N}{x} \left(\ln \frac{N}{x} \right)^{ul/s} dx \asymp N^l(\ln N)^{1+ul/s}. \end{aligned}$$

Let $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k = [b_{i_1, j_1}, b_{i_1, j_1+1}; \dots; b_{i_l, j_l}, b_{i_l, j_l+1}]$. Introduce the functions

$$\begin{aligned} \varphi_{i_1, \dots, i_l; j_1, \dots, j_l}^k(t) &= \\ &= \begin{cases} A_k \frac{((t_1 - b_{i_1, j_1})(b_{i_1, j_1+1} - t_1) \dots (t_l - b_{i_l, j_l})(b_{i_l, j_l+1} - t_l))^s}{(h_k/M_k)^{(2l-1)s}((k+1)/N)^{v\gamma}} \left(1 + \left| \ln^u \left(\frac{k+1}{N} \right)^v \right| \right), \\ t \in \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k, 0, t \in \Omega \setminus \Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k, \end{cases} \end{aligned}$$

$h_k = ((k+1)/N)^v - (k/N)^v$, $k = 0, 1, \dots, N - 1$. Constants A_k , $k = 0, 1, \dots, N - 1$, are chosen such that

$$|D^s \varphi_{i_1, \dots, i_l; j_1, \dots, j_l}^k| \leq \frac{1}{((k+1)/N)^{v\gamma}} \left(1 + \left| \ln^u \left(\frac{k+1}{N} \right)^v \right| \right).$$

Obviously, such constants exist and do not depend on N, u, γ .

Let estimate the maximum values of $\varphi_{i_1, \dots, i_l; j_1, \dots, j_l}^k(t)$. Clearly,

$$\begin{aligned} \varphi_{i_1, \dots, i_l; j_1, \dots, j_l}^k(t) &\geq A_k \left(\frac{h_k}{M_k} \right)^s \left(\frac{N}{k+1} \right)^{v\gamma} \left(1 + \left| \ln^u \left(\frac{k+1}{N} \right)^v \right| \right) = \\ &= A_k \left(\left(\frac{k+1}{N} \right)^v - \left(\frac{k}{N} \right)^v \right)^s \frac{1}{\left((\ln \frac{N}{k})^{u/s} + 1 \right)^s} \left(\frac{N}{k+1} \right)^{v\gamma} \left(1 + \left| \ln^u \left(\frac{k+1}{N} \right)^v \right| \right) \geq \frac{c}{N^s} \end{aligned}$$

for $k = 1, 2, \dots, N - 1$;

$$\varphi_{i_1, \dots, i_l; j_1, \dots, j_l}^0(t) \geq A_0 \left(\frac{h_0}{M_0} \right)^s \left(\frac{N}{1} \right)^{v\gamma} \left(1 + \left| \ln^u \left(\frac{1}{N} \right)^v \right| \right) \geq c \frac{1}{N^s}.$$

Let $\xi(t)$ be a linear combination $\xi(t) = \sum_{k, i_1, \dots, i_l; j_1, \dots, j_l} C_{i_1, \dots, i_l; j_1, \dots, j_l}^k \varphi_{i_1, \dots, i_l; j_1, \dots, j_l}^k(t)$,

where $|C_{i_1, \dots, i_l; j_1, \dots, j_l}^k| \leq 1$. Here the summation is taken over all domains $\Delta_{i_1, \dots, i_l; j_1, \dots, j_l}^k$ of Ω .

Repeating the arguments presented in Anuchina et al. (1979); Babenko (1985); Boykov (1998) we have $\delta_n(Q_{r\gamma}^u, (\Omega, 1)) \geq cn^{-s/l}(\ln n)^{u+s/l}$. \square

Remark. The estimate $d_n(Q_{r\gamma}^u(\Omega, 1), C) \geq cn^{-s/l}(\ln n)^{u+s/l}$ follows from the definition of $\xi(t)$ and Lemma 2.8.

Combining the statements of Theorems 4.4 – 4.6 we have the following

THEOREM 4.7. Let $\Omega = [-1, 1]^l$, $l \geq 2$, $u = 1, 2, \dots$, $v = s/(s - \gamma)$. Then $\delta_n(Q_{r\gamma}^u(\Omega, 1)) \asymp d_n(Q_{r\gamma}^u(\Omega, 1), C) \asymp n^{-s/l}$ if $v < l/(l - 1)$;
 $cn^{-s/l}(\ln n)^{(ul+s)/l} \leq \delta_n(Q_{r\gamma}^u(\Omega, 1)) \leq 2d_n(Q_{r\gamma}^u(\Omega, 1), C) \leq cn^{-s/l}(\ln n)^{us/(r+1-\mu)}$
for $lu/(r + 1 - \mu) \geq ul/s + 1$,
 $\delta_n(Q_{r\gamma}^u(\Omega, 1)) \asymp d_n(Q_{r\gamma}^u(\Omega, 1), C) \asymp n^{-s/l}(\ln n)^{(ul+s)/l}$ for $lu/(r + 1 - \mu) < ul/s + 1$,
if $v = l/(l - 1)$.

THEOREM 4.8. Let $\Omega = [-1, 1]^l$, $l \geq 2$, $v = s/(s - \gamma)$, $v > l/(l - 1)$. Then $\delta_n(Q_{r\gamma}^u(\Omega, 1)) \geq cn^{-(s-\gamma)/(l-1)} \ln^u n$.

Proof. The proof of the theorem is similar to the proof of Theorem 4.2. The difference is that we define the function $\varphi_{i_1, \dots, i_l}^0(t)$ by

$$\varphi_{i_1, \dots, i_l}^0(t) = \begin{cases} A_0 \frac{((t_1 - b_{i_1}^0)(b_{i_1+1}^0 - t_1) \dots (t_l - b_{i_l}^0)(b_{i_l+1}^0 - t_l))^s}{h_0^{(2l-1)s}} N^{v\gamma} \ln^u N, & t \in \Delta_{i_1, \dots, i_l}^0, \\ 0, & t \in \Omega \setminus \Delta_{i_1, \dots, i_l}^0. \end{cases}$$

\square

THEOREM 4.9. Let $\Omega = [-1, 1]^l$, $l \geq 2$, $v = s/(s - \gamma)$, $v > l/(l - 1)$. Then $d_n(Q_{r\gamma}^u(\Omega, 1), C) \leq cn^{-(s-\gamma)/(l-1)} \ln^u n$.

Proof. We decompose the domain Ω in to subdomains $\Delta_{i_1, \dots, i_l}^k$, $k = 0, 1, \dots, N - 1$, following the procedure which was described in the proof of Theorem 4.3.

Clearly, the number of $\Delta_{i_1, \dots, i_l}^k$ is estimated by

$$n \asymp N^{v(l-1)}. \quad (4.31)$$

The polynomial $P_{s, \dots, s}(f; \Delta_{i_1, \dots, i_l}^k)$ interpolates f in $\Delta_{i_1, \dots, i_l}^k$, $k = 0, 1, \dots, N - 1$. Hence the local spline f_N is composed of the polynomials $P_{s, \dots, s}(f; \Delta_{i_1, \dots, i_l}^k)$, $k = 0, 1, \dots, N - 1$.

It is easy to see that for $1 \leq k \leq N - 1$ the following estimate holds

$$\|f - f_N\|_{C(\Delta_{i_1, \dots, i_l}^k)} \leq cN^{-s}(\ln N)^u. \quad (4.32)$$

Indeed $\|f - f_N\|_{C(\Delta_{i_1, \dots, i_l}^k)} \leq ch_k^s \frac{|\ln(\frac{k}{N})|^v|^u}{((k/N)^v)^\gamma} \leq ch_k^s (\frac{N}{k})^{v\gamma} (\ln N)^u = \frac{c}{N^s} (\ln N)^u$.

Let $k = 0$. Without loss of generality we demonstrate our computations in $\Delta_{0, \dots, 0}^0 = [-1, t_1; -1, t_1; \dots; -1, t_1]$, where $t_1 = -1 + (\frac{1}{N})^v$. Using Taylor's expansion (4.6) we obtain

$$\begin{aligned} & \|f - f_N\|_{C(\Delta_{0, \dots, 0}^0)} \leq c\lambda_s^l E_{r, \dots, r}(f, \Delta_{0, \dots, 0}^0) \leq \\ & \leq c \max_{t \in \Delta_{0, \dots, 0}^0} \left| \sum_{|k|=r+1} \frac{1}{k!} \int_0^1 (1-\tau)^r (t_k + 1)^k \frac{(1 + |\ln^u d(-1 + \tau(t_k + 1)), \Gamma)|)}{(d(-1 + \tau(t_k + 1)), \Gamma))^{1-\zeta}} d\tau \right| \leq c \frac{\ln^u N}{N^s}. \end{aligned}$$

From the previous estimate and the equality (4.31) one obtains $\|f - f_N\|_{C(\Omega)} \leq cN^{-s} \ln^u N \leq cn^{-(s-\gamma)/(l-1)} \ln^u n$.

To construct the continuous local spline f_N^* approximating $Q_{r\gamma}^u(\Omega, 1)$ for $v > l/(l-1)$ with the error $c(\ln^u n)n^{-(s-\gamma)/(l-1)}$, we employ all above constructions for the continuous local spline f_N^* approximating $\bar{Q}_{r\gamma}^u(\Omega, 1)$ when $v \geq l/(l-1)$, cf. the proof of 4.3.

Thus, for $0 \leq k \leq N-1$, the following estimates hold

$$\|f - f_N^*\|_{C(\Delta_{i_1, \dots, i_l}^k)} \leq cN^{-s}(\ln N)^u, k = 1, \dots, N-1, \quad (4.33)$$

$$\|f - f_N^*\|_{C(\Delta_{0, \dots, 0}^0)} \leq ch_0^{r+1-\mu} |\ln^u h_0| \leq cN^{-s} \ln^u N. \quad (4.34)$$

From the previous estimates and the equality (4.31) we have $\|f - f_N^*\|_{C(\Omega)} \leq cN^{-s} \ln^u N \leq cn^{-(s-\gamma)/(l-1)} \ln^u n$.

Since the number of the nodes used to construct the continuous local spline f_N^* is s^l in each $\Delta_{i_1, \dots, i_l}^k$, $k = 0, 1, \dots, N-1$, we state the following estimate $d_n(Q_{r\gamma}^u(\Omega, 1), C) \leq cn^{-(s-\gamma)/(l-1)} \ln^u n$. \square

Combining Theorem 4.8 and Theorem 4.9 leads us to the following

THEOREM 4.10. Let $\Omega = [-1, 1]^l$, $l \geq 2$, $u = 1, 2, \dots$, $v = s/(s-\gamma)$, $v > l/(l-1)$. Then $\delta_n(Q_{r\gamma}^u(\Omega, 1)) \asymp d_n(Q_{r\gamma}^u(\Omega, 1), C) \asymp n^{-(s-\gamma)/(l-1)} \ln^u n$.

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